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Families of k -derivations on k -algebras

Philippe Bonnet

Mathematisches Institut, Universität Basel, Rheinsprung 21, 4051 Basel, Switzerland

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Abstract

Let A be an integral k -algebra of finite type over a field k of characteristic zero. Let \mathcal{F} be a family of k -derivations on A and $M_{\mathcal{F}}$ the A -module spanned by \mathcal{F} . In this paper, we generalise a result due to Nowicki and construct an element ∂ of $M_{\mathcal{F}}$ such that $\ker \partial = \bigcap_{d \in \mathcal{F}} \ker d$. Such a derivation is called \mathcal{F} -minimal. Then we establish a density theorem for \mathcal{F} -minimal derivations in $M_{\mathcal{F}}$.

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1. Introduction

Let A be an integral k -algebra of finite type over a field k of characteristic zero. For convenience, we will say that a sub-algebra B of A is *algebraically closed in A* if every element a of A that is algebraic over B belongs to B . Let \mathcal{F} be a family of k -derivations on A . In this paper, we are interested in describing the kernel of this family, i.e. the following set

$$\ker \mathcal{F} = \bigcap_{d \in \mathcal{F}} \ker d.$$

Let $M_{\mathcal{F}}$ be the A -module spanned by the elements of \mathcal{F} . By analogy with the theory of foliations, we say that an element f of A is a *first integral of a k -derivation d* if $d(f) = 0$ and $f \notin k$. Similarly f is a *first integral of \mathcal{F}* if $d(f) = 0$ for every $d \in \mathcal{F}$, and $f \notin k$. First, integrals correspond to the notion of constants for a derivation (see [10]), except that they must not belong to the coefficient field k .

E-mail address: bonnet@math-lab.unibas.ch (P. Bonnet).

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The description of the kernels $\ker \mathcal{F}$ is usually quite tricky because of their complexity. Indeed, since Nagata's works (see [8]), it is well-known that the sub-algebra $B = \ker \mathcal{F}$ need not be finitely generated. Nagata's construction uses locally nilpotent derivations on a k -algebra A of Krull dimension $n \geq 32$. This result has been refined by Deveney and Finston, who constructed a locally nilpotent k -derivation on $k[x_1, \dots, x_7]$ whose kernel is not finitely generated (see [2]). Recently this result has been improved by Daigle and Freudenburg (see [1]), with an example of a locally nilpotent derivation on $k[x_1, \dots, x_5]$ having as kernel a non-finitely generated algebra. In contrast, such behaviours do not occur in low dimensions. For instance, derivations on $k[x_1, \dots, x_n]$ have as kernel a finitely generated k -algebra if $n \leq 3$ (see [9]).

In what follows, we will choose to express $\ker \mathcal{F}$ not as a k -algebra, but in terms of the derivations involved in its construction. Our starting point is an article of Nowicki (see [10]) where he proved the following two theorems.

Theorem 1.1 (Nowicki [10]). *Let A be an integral k -algebra of finite type over a field k of characteristic zero. Let \mathcal{F} be a family of k -derivations on A . Then there exists a k -derivation d on A such that $\ker d = \ker \mathcal{F}$.*

Theorem 1.2 (Nowicki [10]). *Let A be an integral k -algebra of finite type over a field k of characteristic zero. Let B be a sub-algebra of A . Then B is algebraically closed in A if and only if B is the kernel of a k -derivation on A .*

The proof of Theorem 1.1 is very elegant and uses Noether normalisation lemma. However the construction of the derivation d is independent of the family \mathcal{F} , and it only uses the fact that the ring $B = \ker \mathcal{F}$ is algebraically closed in A . In this paper we will refine Theorem 1.1, and we will express the derivation d in terms of the elements of \mathcal{F} . More precisely.

Theorem 1.3. *Let A be an integral k -algebra of finite type over a field k of characteristic zero. Let $\mathcal{F} = \{d_i\}_{i \in I}$ be a family of k -derivations on A . Then there exists an A -linear combination $d = \sum a_i d_i$ such that $\ker d = \ker \mathcal{F}$.*

A k -derivation d in $M_{\mathcal{F}}$ is \mathcal{F} -minimal if $\ker d = \ker \mathcal{F}$. Here \mathcal{F} -minimality means that the kernel of d is smallest among all kernels of k -derivations in $M_{\mathcal{F}}$. We denote by $M_{\mathcal{F}, \min}$ the set of all \mathcal{F} -minimal k -derivations on A . In terms of first integrals, Theorem 1.3 can be reinterpreted as follows:

Corollary 1.4. *Let A be an integral k -algebra of finite type over a field k of characteristic zero. Let $\mathcal{F} = \{d_i\}_{i \in I}$ be a family of k -derivations on A . If every A -linear combination $d = \sum a_i d_i$ admits a first integral, then \mathcal{F} admits a first integral.*

Let $d = \sum a_i d_i$ be an A -linear combination of elements of \mathcal{F} . A priori its first integrals (if any) depend on the coefficients a_i . But if every such combination has a first integral, then the previous corollary asserts that we can choose a first integral f that is independent of the a_i . In particular $d_i(f) = 0$ for any $i \in I$.

Definition 1.5. Let k be an algebraically closed field of characteristic zero. Let E be a k -vector space and Ω a subset of E . The set Ω is residual in E if, for any finite dimensional k -subspace F of E , $\Omega \cap F$ is a countable intersection of Zariski open sets of F (possibly empty).

Note that if $k = \mathbb{C}$ and $\Omega \cap F \neq \emptyset$, then $\Omega \cap F$ is dense in F for the Zariski and metric topologies on F . This latter assertion is based on Baire's theorem about countable intersection of dense open sets in a complete space, and also on the fact that every non-empty Zariski open set is dense in F for the metric topology. So residuality can be seen as a version of density adapted to infinite dimensional spaces. With this definition, Theorem 1.3 yields the following result.

Theorem 1.6. Let k be an algebraically closed field of characteristic zero. Let A be an integral k -algebra of finite type. Let \mathcal{F} be a family of k -derivations on A . Then $M_{\mathcal{F}, \min}$ is a non-empty residual subset of $M_{\mathcal{F}}$.

At the end of this paper, we will give a proof of Theorem 1.2 based on Theorem 1.3, and we will illustrate the notion of residuality with an example. Note that in this paper, we have not investigated the field of rational first integrals, i.e. the elements f of the fraction field $K(A)$ of A such that $d(f) = 0$ for any d in \mathcal{F} . This field can be extremely large compared to the kernel of \mathcal{F} , as is the case for the Euler vector field:

$$d = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$$

on k^n , but it can also be reduced to the field k (see for instance [11] or [5]). One question could be to try and find an analogue to Theorem 1.3. More precisely, given a collection of k -derivations $\{d_i\}$ on a field K of finite transcendence degree, with $L = \bigcap \ker d_i | K$, does there exist a K -linear combination d of the d_i having L as its kernel?

2. Reduction to a couple of derivations

Throughout this paper, A will be an integral k -algebra of finite type over a field k of characteristic zero. Let \mathcal{F} be a family of k -derivations on A . We will say that the triplet (k, A, \mathcal{F}) enjoys the property \mathcal{P} if there exists an A -linear combination $d = \sum a_i d_i$ of elements d_i of \mathcal{F} such that

$$\ker d = \ker \mathcal{F}.$$

In this section, we are going to see how to restrict the proof of Theorem 1.3 to the case of a couple of derivations enjoying some remarkable properties. More precisely:

Proposition 2.1. \mathcal{P} holds for any triplet (k, A, \mathcal{F}) if and only if \mathcal{P} holds for any triplet (k', A', \mathcal{F}') , where $\mathcal{F}' = \{d'_1, d'_2\}$ is a couple of k' -derivations satisfying the two conditions: (1) $\ker d'_1 \cap \ker d'_2 = k'$ and (2) there exist two elements x_1, x_2 of A' such that $d'_1(x_i) = 1$ and $d'_i(x_j) = 0$ if $i \neq j$.

The proof of this proposition is a consequence of the following lemmas.

Lemma 2.2. \mathcal{P} holds for any triplet (k, A, \mathcal{F}) if and only if it holds for any (k', A', \mathcal{F}') , where \mathcal{F}' is a finite family.

Proof. One direction is clear. For the other, consider the triplet (k, A, \mathcal{F}) and let $\text{Der}_k(A, A)$ be the A -module of k -derivations on A . Since A is a finite k -algebra, $\text{Der}_k(A, A)$ is a noetherian A -module. So the A -submodule $M_{\mathcal{F}}$ of $\text{Der}_k(A, A)$ spanned by the elements of \mathcal{F} is finitely generated. Let $\mathcal{F}' = \{d_1, \dots, d_r\}$ be a finite subset of \mathcal{F} whose elements span the A -module $M_{\mathcal{F}}$. Then we have:

$$\ker \mathcal{F} = \ker \mathcal{F}' = \ker d_1 \cap \dots \cap \ker d_r.$$

The inclusion $\ker \mathcal{F} \subseteq \ker \mathcal{F}'$ is obvious. Conversely let f be an element of A such that $d_1(f) = \dots = d_r(f) = 0$. For any k -derivation d of \mathcal{F} , there exist some elements a_1, \dots, a_r of A such that:

$$d = a_1 d_1 + \dots + a_r d_r.$$

Therefore $d(f) = a_1 d_1(f) + \dots + a_r d_r(f) = 0$, f belongs to $\ker \mathcal{F}$ and $\ker \mathcal{F} = \ker \mathcal{F}'$. Assume that \mathcal{P} holds for any (k', A', \mathcal{F}') , where \mathcal{F}' is finite. Apply this property to the triplet (k, A, \mathcal{F}') . Then there exists an A -linear combination \hat{d} of elements of \mathcal{F}' such that $\ker \hat{d} = \ker \mathcal{F}'$. Since $\ker \mathcal{F}' = \ker \mathcal{F}$ and \mathcal{F}' is a subset of \mathcal{F} , the result follows. \square

Lemma 2.3. \mathcal{P} holds for any triplet (k, A, \mathcal{F}) , where \mathcal{F} is finite, if and only if it holds for any (k', A', \mathcal{F}') , where $\mathcal{F}' = \{d_1, d_2\}$.

Proof. One direction is clear. The other will be proved by induction on the order r of \mathcal{F} . If $r = 1$, then \mathcal{F} consists of one derivation d_1 and we choose $\hat{d} = d_1$. If $r = 2$, then \mathcal{F} is a couple of derivations and the result follows by assumption. Assume the property holds to the order $r \geq 2$, and let (k, A, \mathcal{F}) be a triplet such that $\mathcal{F} = \{d_1, \dots, d_{r+1}\}$. By assumption there exists a k -derivation $\hat{d}' = a_1 d_1 + a_2 d_2$, where every a_i belongs to A , such that:

$$\ker \hat{d}' = \ker d_1 \cap \ker d_2.$$

Consider the family $\{\hat{d}', d_3, \dots, d_{r+1}\}$. Since the property holds to the order r , there exists a k -derivation $\hat{d} = b' \hat{d}' + \dots + b_{r+1} d_{r+1}$, where b', b_3, \dots, b_{r+1} belong to A , such that

$$\ker \hat{d} = \ker \hat{d}' \cap \ker d_3 \cap \dots \cap \ker d_{r+1} = \ker d_1 \cap \ker d_2 \cap \dots \cap \ker d_{r+1}.$$

Since \hat{d}' is a A -linear combination of d_1, d_2 , \hat{d} belongs to the A -module spanned by d_1, \dots, d_{r+1} , and the result follows. \square

Lemma 2.4. \mathcal{P} holds for any triplet (k, A, \mathcal{F}) , where $\mathcal{F} = \{d_1, d_2\}$, if and only if it holds for any (k', A', \mathcal{F}') , where $\mathcal{F}' = \{d'_1, d'_2\}$ is a couple of k' -derivations for which there exist two elements x_1, x_2 of A' such that $d'_i(x_i) = 1$ and $d'_i(x_j) = 0$ if $i \neq j$.

Proof. One direction is clear. For the other, assume that \mathcal{P} holds for any (k', A', \mathcal{F}') , where $\mathcal{F}' = \{d'_1, d'_2\}$ is a couple of k' -derivations for which there exist two elements x_1, x_2

of A' such that $d'_i(x_i) = 1$ and $d'_i(x_j) = 0$ if $i \neq j$. Let (k, A, \mathcal{F}) be a triplet for which $\mathcal{F} = \{d_1, d_2\}$. If $\ker d_1 \subseteq \ker d_2$ or $\ker d_2 \subseteq \ker d_1$, then we have:

$$\ker \mathcal{F} = \ker d_1 \quad \text{or} \quad \ker \mathcal{F} = \ker d_2$$

and \mathcal{P} holds by choosing either $\partial = d_1$ or $\partial = d_2$. So we may assume that:

$$\ker d_2 \not\subseteq \ker d_1 \quad \text{and} \quad \ker d_1 \not\subseteq \ker d_2.$$

By assumption, there exist two elements x_1, x_2 of A such that:

$$d_1(x_1) \neq 0, \quad d_2(x_1) = 0, \quad d_2(x_2) \neq 0, \quad d_1(x_2) = 0.$$

We set $p = d_1(x_1)d_2(x_2)$ and consider the triplet

$$(k', A', \mathcal{F}') = \left(k', A' \left[\frac{1}{p} \right], \left\{ \frac{d_1}{d_1(x_1)}, \frac{d_2}{d_2(x_2)} \right\} \right).$$

By construction A' is a k' -algebra of finite type, and it is a domain. Moreover d'_1, d'_2 act on A' as k' -derivations, $d'_i(x_i) = 1$ and $d'_i(x_j) = 0$ if $i \neq j$. So there exists an A' -linear combination ∂' of d'_1, d'_2 such that

$$\ker \mathcal{F}' = \ker \partial'.$$

Up to replacing ∂' by $p^n \partial'$ for n big enough, we may assume that ∂' is an A -linear combination of d_1, d_2 . Let ∂ be the restriction of ∂' to A . Let us prove that $\ker \mathcal{F} = \ker \partial$.

The inclusion $\ker \mathcal{F} \subseteq \ker \partial$ is clear. Now let x be an element of A such that $\partial(x) = 0$. Then $\partial'(x) = 0$ in A' , and x belongs to $\ker d'_1 \cap \ker d'_2 \cap A$. Since d_1 (resp. d_2) is proportional to d'_1 (resp. d'_2), we find

$$d_1(x) = d_2(x) = 0.$$

Since x belongs to A , x belongs to $\ker d_1 \cap \ker d_2 = \ker \mathcal{F}$ and the result follows. \square

Lemma 2.5. *The following assertions are equivalent:*

- \mathcal{P} holds for any triplet (k, A, \mathcal{F}) , where $\mathcal{F} = \{d_1, d_2\}$ is a couple of k -derivations for which there exist two elements x_1, x_2 of A such that $d_i(x_i) = 1$ and $d_i(x_j) = 0$ if $i \neq j$.
- \mathcal{P} holds for any triplet (k', A', \mathcal{F}') , where $\mathcal{F}' = \{d'_1, d'_2\}$ is a couple of k' -derivations satisfying the two conditions: (1) $\ker d'_1 \cap \ker d'_2 = k'$ and (2) there exist two elements x_1, x_2 of A' such that $d'_i(x_i) = 1$ and $d'_i(x_j) = 0$ if $i \neq j$.

Proof. The first assertion implies clearly the second. Assume now that the second holds, and let (k, A, \mathcal{F}) be a triplet satisfying the conditions of the first assertion. Let k' be the fraction field of $\ker \mathcal{F}$, and consider the following triplet:

$$(k', A', \mathcal{F}') = (k', A \otimes_k k', \{d'_1, d'_2\}),$$

where every d'_i acts on $A \otimes_k k'$ according to the following rule:

$$d'_i(x \otimes f) = d_i(x) \otimes f.$$

If $S = \ker \mathcal{F} - \{0\}$, then $A' = A_S$ and A' is a domain. Since k' contains k , A' is a k' -algebra of finite type. Moreover $d'_i(a/b) = d_i(a)/b$ for any a/b in A_S . In particular, every d'_i is a k' -derivation. Let $f = a/b$ be an element of A_S such that $d'_1(f) = d'_2(f) = 0$. By construction we get $d_1(a) = d_2(a) = 0$, a belongs to $\ker \mathcal{F}$ and f lies in k' . Thus we have

$$\ker \mathcal{F}' = \ker d'_1 \cap \ker d'_2 = k'.$$

So the conditions of the second assertion hold, and there exists an A_S -linear combination $\partial' = a'_1 d'_1 + a'_2 d'_2$ such that

$$\ker \partial' = \ker \mathcal{F}' = k'.$$

Up to a multiplication by an element of S , we may assume that a'_1 and a'_2 belong to A . Denote by ∂ the derivation $a'_1 d_1 + a'_2 d_2$. Let f be an element of $\ker \partial$. Since A is contained in A_S and every d'_i extends d_i to A_S , we have $\partial(f) = \partial'(f) = 0$. So $d_i(f) = d'_i(f) = 0$ for $i = 1, 2$ and f belongs to $\ker \mathcal{F}$. Therefore we get

$$\ker \partial = \ker \mathcal{F} = \ker d_1 \cap \ker d_2. \quad \square$$

3. Passage to a complete regular local ring

Let A be an integral k -algebra of finite type, and $\mathcal{F} = \{d_1, d_2\}$ a couple of k -derivations satisfying the conditions of Proposition 2.1, i.e. (1) $\ker d_1 \cap \ker d_2 = k$ and (2) there exist two elements x_1, x_2 of A such that $d_i(x_i) = 1$ for all i and $d_i(x_j) = 0$ if $i \neq j$. In this section, we will see how to extend the k -derivations d_1, d_2 into a couple of L -derivations on a formal ring $L[[t_1, \dots, t_n]]$. This will enable us to rewrite these derivations into a canonical form, that will prove easier to handle.

Proposition 3.1. *Let A be an integral k -algebra of finite type, and $\mathcal{F} = \{d_1, d_2\}$ be a couple of k -derivations satisfying the conditions of Proposition 2.1. Let x_1, x_2 be two elements of A such that $d_i(x_i) = 1$ for all i and $d_i(x_j) = 0$ if $i \neq j$. Then there exist two elements λ_1, λ_2 of k , and an extension L of k such that*

- A is a subring of $L[[t_1, \dots, t_n]]$,
- d_1, d_2 extend to L -derivations on $L[[t_1, \dots, t_n]]$,
- $x_1 - \lambda_1, x_2 - \lambda_2$ belong to the maximal ideal of $L[[t_1, \dots, t_n]]$.

The proof of this proposition will split into several lemmas.

Lemma 3.2. *Let x_1, x_2 be two elements of A satisfying the conditions of Proposition 3.1. Then x_1, x_2 are algebraically independent in the k -algebra A .*

Proof. Assume there exists a non-zero polynomial P in $k[u, v]$ such that $P(x_1, x_2) = 0$. We choose P of minimal homogeneous degree with respect to u, v . Since $d_i(x_i) = 1$ for all

i and $d_i(x_j) = 0$ if $i \neq j$, we get by derivation

$$d_1(P(x_1, x_2)) = \frac{\partial P}{\partial u}(x_1, x_2) = 0 \quad \text{and} \quad d_2(P(x_1, x_2)) = \frac{\partial P}{\partial v}(x_1, x_2) = 0.$$

By minimality of the degree, this implies that $\partial P / \partial u = \partial P / \partial v = 0$. Therefore P is constant and $P(x_1, x_2) = 0$ implies that $P = 0$, hence a contradiction. \square

Lemma 3.3. *There exist two elements λ_1, λ_2 of k , and a maximal ideal \mathcal{M} of A such that $x_1 - \lambda_1, x_2 - \lambda_2$ belong to \mathcal{M} and $A_{\mathcal{M}}$ is a regular local ring.*

Proof. Up to localizing A with respect to a non-zero element g of A , we may assume that $\Omega_{A_{(g)}/k} \simeq (\Omega_{A/k})_{(g)}$ is free. By generic smoothness (see [3]), $A' = A_{(g)}$ is regular over every maximal ideal \mathcal{M}' not containing g . Since x_1, x_2 are algebraically independent, the inclusion induces an injective map

$$L : k[u, v] \longrightarrow A', \quad P \longmapsto P(x_1, x_2).$$

Therefore the map $L^* : \text{Spec}(A') \rightarrow \text{Spec}(k[u, v])$ is dominant, and there exists an element $f \neq 0$ of $k[u, v]$ such that every fibre $L^{-1}(\mathcal{P})$ is non-empty for any maximal ideal \mathcal{P} of $k[u, v]$ not containing f . Since f is non-zero, there exists a couple (λ_1, λ_2) in k^2 such that $f(\lambda_1, \lambda_2) \neq 0$. Consider the ideal

$$\mathcal{P} = (u - \lambda_1, v - \lambda_2).$$

By construction, \mathcal{P} is maximal in $k[u, v]$ and does not contain f . So the fibre $L^{-1}(\mathcal{P})$ is not empty. In particular, it contains a maximal ideal \mathcal{M}' of A' . If \mathcal{M} denotes the intersection $\mathcal{M}' \cap A$, then \mathcal{M} is a maximal ideal not containing g , and we have the isomorphism of k -algebras

$$A'_{\mathcal{M}'} \simeq A_{\mathcal{M}}.$$

Since A' is regular over every maximal ideal, $A_{\mathcal{M}}$ is a regular local ring. By construction \mathcal{M} contains $x_1 - \lambda_1, x_2 - \lambda_2$, and the result follows. \square

Since λ_1, λ_2 are annihilated by the d_i , we may replace x_i by $x_i - \lambda_i$ without changing the conditions at the beginning of this section. So we may assume that x_1, x_2 belong to a maximal ideal \mathcal{M} of A such that the k -algebra $A_{\mathcal{M}}$ is regular. By an easy computation, we get for any positive integer r :

$$d_1(\mathcal{M}^r) \subseteq \mathcal{M}^{r-1} \quad \text{and} \quad d_2(\mathcal{M}^r) \subseteq \mathcal{M}^{r-1}.$$

Therefore d_1 and d_2 are continuous on $A_{\mathcal{M}}$ for the \mathcal{M} -adic topology, and they uniquely extend into a couple of k -derivations on the \mathcal{M} -adic completion R of $A_{\mathcal{M}}$. We still denote by d_i this extension. Since $A_{\mathcal{M}}$ is regular and contains the field k , by Cohen structure theorem (see [3]), there exists an extension L of k such that

$$R \simeq L[[t_1, \dots, t_n]],$$

where n is the Krull dimension of A . So the d_i can be viewed as k -derivations on $L[[t_1, \dots, t_n]]$. In order to get Proposition 3.1, we only need to check that:

Lemma 3.4. d_1 and d_2 are L -derivations on $L[[t_1, \dots, t_n]]$.

Proof. It suffices to prove that $d_i(L) = \{0\}$ for $i = 1, 2$. First note that L is isomorphic to A/\mathcal{M} . Since A is a finite k -algebra, the field L is also a finite k -algebra. So L is a finite extension of k (see [4]). Let ζ be an element of L , and let P be a polynomial in $k[t]$ of minimal degree such that $P(\zeta) = 0$. By derivation, we get

$$d_i(P(\zeta)) = P'(\zeta)d_i(\zeta) = 0.$$

By minimality of the degree, $P'(\zeta) \neq 0$ and $d_i(\zeta) = 0$. Since this holds for any ζ in L , the result follows. \square

4. Canonical form for a couple of L -derivations

In this section, we consider a couple of L -derivations d_1, d_2 on $L[[t_1, \dots, t_n]]$, satisfying the following condition: there exist two elements x_1, x_2 of $L[[t_1, \dots, t_n]]$ such that $d_i(x_i) = 1$ for $i = 1, 2$ and $d_i(x_j) = 0$ if $i \neq j$. We are going to search for a system of parameters for which these derivations look simpler. Recall that a system of parameters is a family of formal functions s_1, \dots, s_n generating the maximal ideal (t_1, \dots, t_n) of $L[[t_1, \dots, t_n]]$.

Lemma 4.1. Let d be an L -derivation on $L[[t_1, \dots, t_n]]$, where $n > 1$. Assume there exists a formal function x_1 such that $d(x_1) = 1$. Then there exists some formal functions y_2, \dots, y_n such that x_1, y_2, \dots, y_n is a system of parameters and $d(y_i) = 0$ for any i . In particular $d = \partial/\partial x_1$ in this system of parameters.

Proof. Let \mathcal{M} be the maximal ideal of $L[[t_1, \dots, t_n]]$. Since d is an L -derivation, we have $d(\mathcal{M}^2) \subseteq \mathcal{M}$. So x_1 belongs to $\mathcal{M} - \mathcal{M}^2$ because $d(x_1) = 1$. Let y_2^0, \dots, y_n^0 be a system of formal functions such that x_1, y_2^0, \dots, y_n^0 form a basis of $\mathcal{M}/\mathcal{M}^2$. By Nakayama Lemma, x_1, y_2^0, \dots, y_n^0 is a system of parameters. If $\lambda_i = d(y_i)(0)$ and $y_i^1 = y_i^0 - \lambda_i x_1$, then we find:

$$d(y_i^1)(0) = \lambda_i - \lambda_i d(x_1)(0) = 0$$

and x_1, y_2^1, \dots, y_n^1 is again a system of parameters. For any integer $k > 0$, we are going to construct a system y_2^k, \dots, y_n^k of formal functions satisfying the following conditions:

- x_1, y_2^k, \dots, y_n^k is a system of parameters,
- for any i , $y_i^{k+1} - y_i^k \equiv 0[\mathcal{M}^{k+1}]$,
- for any i , $d(y_i^k) \equiv 0[\mathcal{M}^k]$.

Assume for the moment that such a construction is possible. Then for any i , the sequence $(y_i^k)_{k>0}$ is Cauchy for the \mathcal{M} -adic topology on $L[[t_1, \dots, t_n]]$. Since $L[[t_1, \dots, t_n]]$ is complete, $(y_i^k)_{k>0}$ converges to a formal function y_i . By construction, x_1, y_2, \dots, y_n span

the vector space $\mathcal{M}/\mathcal{M}^2$, hence it is a system of parameters by Nakayama Lemma. Moreover by passing to the limit, we find

$$\forall i > 1, \quad d(y_i) = 0$$

and the result follows. We proceed to this construction by induction on $k > 0$. The case $k = 1$ has already been treated above. Assume the construction holds up to the order k . Since $d(y_i^k) \equiv 0 [\mathcal{M}^k]$, there exists a homogeneous polynomial $Q_{i,k}(u_1, \dots, u_n)$ of degree k such that

$$d(y_i^k) \equiv Q_{i,k}(x_1, y_2^1, \dots, y_n^1) [\mathcal{M}^{k+1}].$$

We set $y_i^{k+1} = y_i^k + P_{i,k}(x_1, y_2^1, \dots, y_n^1)$, where every $P_{i,k}(u_1, \dots, u_n)$ is defined as

$$P_{i,k}(x_1, y_2^1, \dots, y_n^1) = - \int_0^{x_1} Q_{i,k}(u, y_2^1, \dots, y_n^1) du.$$

Since every $Q_{i,k}$ is homogeneous of degree k , every $P_{i,k}$ is homogeneous of degree $k+1$. So $y_i^{k+1} - y_i^k \equiv 0 [\mathcal{M}^{k+1}]$ for any i . In particular, $y_i^{k+1} - y_i^1 \equiv 0 [\mathcal{M}^2]$ for any i , and $x_1, y_2^{k+1}, \dots, y_n^{k+1}$ span the vector space $\mathcal{M}/\mathcal{M}^2$. By Nakayama Lemma, $x_1, y_2^{k+1}, \dots, y_n^{k+1}$ is a system of parameters. Moreover we get by derivation

$$d(y_i^{k+1}) = d(y_i^k) + \frac{\partial P_{i,k}}{\partial u_1}(x_1, y_2^1, \dots, y_n^1) + \sum_{i=2}^n \frac{\partial P_{i,k}}{\partial u_i}(x_1, y_2^1, \dots, y_n^1) d(y_i^1).$$

By construction $\partial P_{i,k}/\partial u_i(x_1, y_2^1, \dots, y_n^1)$ belongs to \mathcal{M}^k and $d(y_i^1)$ belongs to \mathcal{M} . By reduction modulo \mathcal{M}^{k+1} and construction of $P_{i,k}$, we obtain:

$$\begin{aligned} d(y_i^{k+1}) &\equiv d(y_i^k) + \frac{\partial P_{i,k}}{\partial u_1}(x_1, y_2^1, \dots, y_n^1) \equiv Q_{i,k}(x_1, y_2^1, \dots, y_n^1), \\ &+ \frac{\partial P_{i,k}}{\partial u_1}(x_1, y_2^1, \dots, y_n^1) \equiv 0 [\mathcal{M}^{k+1}], \end{aligned}$$

thus ending the construction to the order $k+1$, and the result follows. \square

Proposition 4.2. *Let d_1, d_2 be a couple of L -derivations on $L[[t_1, \dots, t_n]]$, where $n > 2$. Assume there exist two formal functions x_1, x_2 such that $d_i(x_i) = 1$ for any i and $d_i(x_j) = 0$ for $i \neq j$. Then there exist some formal functions y_3, \dots, y_n and a_3, \dots, a_n such that:*

- $x_1, x_2, y_3, \dots, y_n$ is a system of parameters,
- $d_1 = \partial/\partial x_1$ and $d_2 = \partial/\partial x_2 + x_1 \sum_{i \geq 2} a_i \partial/\partial y_i$.

Proof. Since $d_1(x_1) = 1$, Lemma 4.1 asserts there exist some formal functions y'_2, \dots, y'_n such that x_1, y'_2, \dots, y'_n is a system of parameters and $d_1(y'_i) = 0$ for all i . In particular, in this system of parameters, we have

$$d_1 = \frac{\partial}{\partial x_1}.$$

A fortiori, this implies that $\ker d_1 = L[[y'_2, \dots, y'_n]]$. Since $d_2(x_1) = 0$, there exist some formal functions b'_2, \dots, b'_n such that:

$$d_2 = \sum_{i=2}^n b'_i \frac{\partial}{\partial y'_i}.$$

Set $b_i(y'_2, \dots, y'_n) = b'_i(0, y'_2, \dots, y'_n)$ for all i , and consider the L -derivation ∂ :

$$\partial = \sum_{i=2}^n b_i \frac{\partial}{\partial y'_i}.$$

By construction, ∂ acts on $L[[y'_2, \dots, y'_n]]$ and $\partial(f) = d_2(f)(0, y'_2, \dots, y'_n)$ for any formal function f . Since $d_1(x_2) = 0$ and $d_2(x_2) = 1$, x_2 belongs to $L[[y'_2, \dots, y'_n]]$ and $\partial(x_2) = 1$. By Lemma 4.1 applied to ∂ and $L[[y'_2, \dots, y'_n]]$, there exist some formal functions y_3, \dots, y_n in $L[[y'_2, \dots, y'_n]]$ such that

- x_2, y_3, \dots, y_n is a system of parameters of $L[[y'_2, \dots, y'_n]]$,
- for any $i > 2$, $\partial(y_i) = 0$.

By construction $L[[x_2, y_3, \dots, y_n]] = L[[y'_2, \dots, y'_n]]$ and $x_1, x_2, y_3, \dots, y_n$ is a system of parameters of $L[[t_1, \dots, t_n]]$. Since $\ker d_1 = L[[y'_2, \dots, y'_n]]$, this implies

$$d_1(x_2) = d_1(y_3) = \dots = d_1(y_n) = 0.$$

In this system of parameters, the derivation d_1 can be written as

$$d_1 = \frac{\partial}{\partial x_1}.$$

Now $d_2(x_1) = 0$ and $d_2(x_2) = 1$, so that d_2 can be written as

$$d_2 = \frac{\partial}{\partial x_2} + \sum_{i>2} \alpha_i \frac{\partial}{\partial y_i},$$

where all the α_i are formal functions. Since $d_2(y_i) \equiv \partial(y_i) \equiv 0[x_1]$ for any i , every function α_i is divisible by x_1 . Write $\alpha_i = x_1 a_i$, where every a_i belongs to $L[[t_1, \dots, t_n]]$. By construction, we find

$$d_2 = \frac{\partial}{\partial x_2} + x_1 \sum_{i>2} a_i \frac{\partial}{\partial y_i}$$

and the result follows. \square

5. Properties of the derivation δ_m

In this section, we will analyse the properties of some derivations, and we will use them to produce the longly awaited derivation of Theorem 1.3. Let R be a commutative domain

with identity. For any positive integer m , we define the R -derivation δ_m on $R[x_1, x_2]$ as

$$\delta_m = x_1^m \frac{\partial}{\partial x_1} + x_2^m \frac{\partial}{\partial x_2}.$$

Lemma 5.1. *For any $m > 0$, we have $\ker \delta_m = R$.*

Proof. Note that δ_m is homogeneous with respect to the variables x_1, x_2 . So every element of its kernel can be uniquely written as a sum of homogeneous elements each belonging to $\ker \delta_m$. Considering an element of $\ker \delta_m$, we may therefore assume that it is homogeneous with respect to x_1, x_2 .

Let P be an homogeneous element of $\ker \delta_m$. If $m = 1$, then Euler's formula asserts that $\delta_1(P) = \deg(P)P$. In this case $\deg(P) = 0$ or $P = 0$. So P belongs to R and $\ker \delta_1 = R$. Assume now that $m > 1$, and write P as

$$P(x_1, x_2) = \sum_{k=0}^n P_k(x_1)x_2^k,$$

where $P_n(x_1) \neq 0$. Then $\delta_m(P)$ can be written as

$$\delta_m(P) = \sum_{k=0}^n k P_k(x_1)x_2^{k+m-1} + \sum_{k=0}^n x_1^m P'_k(x_1)x_2^k.$$

If $n \neq 0$, then the leading term of $\delta_m(P)$ with respect to x_2 is equal to $n P_n(x_1)x_2^{n+m-1}$ because $m > 1$. In particular $\delta_m(P) \neq 0$ if $n \neq 0$. So n must be equal to 0 and $P = P_0(x_1)$. But then $\delta_m(P) = x_1^m P'_0(x_1) = 0$, which implies that P_0 belongs to R , and the result follows. \square

Lemma 5.2. *Let P, Q be two elements of $R[x_1, x_2]$, where Q is homogeneous of degree k with respect to x_1, x_2 . Assume that they satisfy the following relation:*

$$\delta_m(P) + x_1 x_2^m Q = 0.$$

If $m \geq k + 4$, then P belongs to R and $Q = 0$.

Proof. Up to adding a constant to P , we may assume that $P(0, 0) = 0$. We are going to prove that $P = Q = 0$. By the previous lemma, δ_m is injective on the polynomials with no constant terms. Since Q and δ_m are homogeneous, P needs to be homogeneous. We write P as the sum

$$P(x_1, x_2) = \sum_{i+j=r} a_{i,j} x_1^i x_2^j.$$

By derivation, we find

$$\delta_m(P) = \sum_{i+j=r} i a_{i,j} x_1^{i+m-1} x_2^j + \sum_{i+j=r} j a_{i,j} x_1^i x_2^{j+m-1} = -x_1 x_2^m Q(x_1, x_2).$$

Since Q is homogeneous of degree k , we have $r - 2 = k$ and $r \leq m - 2$, since $m \geq k + 4$. We have $m - 1 > 0$ and the following congruence holds modulo x_2^{m-1} :

$$\delta_m(P) \equiv \sum_{i+j=r} i a_{i,j} x_1^{i+m-1} x_2^j \equiv 0 [x_2^{m-1}].$$

In this sum, all the indices j in this sum satisfy $\leq r \leq m - 2$, so that $i a_{i,j} = 0$ for every couple (i, j) . Therefore $a_{i,j} = 0$ or $i = 0$, and P reduces to a polynomial of the form ax_2^r , where a belongs to R . But then we find

$$\delta_m(P) = arx_2^{m+r-1} = -x_1x_2^m Q(x_1, x_2),$$

which is impossible unless $P = Q = 0$. \square

6. Proof of the main theorem

In this section, we work under the assumptions of Section 3. Let A be an integral k -algebra of finite type over a field of characteristic zero. Let d_1, d_2 be a couple of k -derivations on A , and x_1, x_2 be two elements of A such that $d_i(x_i) = 1$ for any i and $d_i(x_j) = 0$ for $i \neq j$. We assume that $k = \ker d_1 \cap \ker d_2$. According to Section 3, A is embedded in $L[[t_1, \dots, t_n]]$, where L is a finite extension of k , and the d_i extend to L -derivations on $L[[t_1, \dots, t_n]]$. By Proposition 4.2, there exist some formal elements $y_3, \dots, y_n, a_3, \dots, a_n$ of $L[[t_1, \dots, t_n]]$ such that:

- $x_1, x_2, y_3, \dots, y_n$ is a system of parameters,
- $d_1 = \partial/\partial x_1$ and $d_2 = \partial/\partial x_2 + x_1 \sum_{i \geq 3} a_i \partial/\partial y_i$.

For any positive integer m , we consider the following k -derivation Δ_m :

$$\Delta_m = x_1^m d_1 + x_2^m d_2.$$

In order to prove Theorem 1.3, we are going to establish the following result.

Proposition 6.1. *Under the previous assumptions, there exists a positive integer m_0 such that, for any $m \geq m_0$, we have $\ker \Delta_m = \ker d_1 \cap \ker d_2 = k$ in A .*

Before giving the proof, we make explicit the number m_0 . Let R be the ring $L[[y_3, \dots, y_n]]$. We provide $R[[x_1, x_2]] = L[[t_1, \dots, t_n]]$ with the homogeneous degree \deg on the variables x_1, x_2 . More precisely $\deg(x_1) = \deg(x_2) = 1$ and $\deg(y_i) = 0$ for any $i \geq 3$. Let a_i^k be the homogeneous part of a_i of degree k . We denote by ∂_k the L -derivation on $L[[t_1, \dots, t_n]]$:

$$\partial_k = \sum_{i \geq 3} a_i^k \frac{\partial}{\partial y_i}.$$

Every ∂_k is an L -linear operator of degree k on $R[[x_1, x_2]]$, and we have by construction

$$d_2 = \frac{\partial}{\partial x_2} + x_1 \sum_{k \geq 0} \partial_k.$$

Let $\mathcal{D} = \text{Der}_L(L[[t_1, \dots, t_n]], L[[t_1, \dots, t_n]])$ be the space of L -derivations on $L[[t_1, \dots, t_n]]$, and let M be the sub- $R[[x_1, x_2]]$ -module of \mathcal{D} spanned by the ∂_k . Since \mathcal{D} is noetherian, there exists an integer m'_0 such that

$$M = R[[x_1, x_2]]\{\partial_0, \dots, \partial_{m'_0}\}.$$

We denote by m_0 the integer $m_0 = m'_0 + 4$.

Proof of Proposition 6.1. Let f be a non-zero element of $R[[x_1, x_2]]$ such that $\Delta_m(f) = 0$, and assume that $m \geq m_0$. Let us prove by contradiction that $d_1(f) = d_2(f) = 0$. We decompose f into its sum of homogeneous parts with respect to \deg :

$$f = f_0 + f_1 + \dots + f_k + \dots.$$

Assertion 1: $f_0 \neq 0$. Let i be the smallest index such that $f_i \neq 0$. Since $\Delta_m(f) = 0$, we have $\delta_m(f) + x_1 x_2^m \sum_{k \geq 0} \partial_k(f) = 0$. By considering only the terms of degree $i + m - 1$ in this equation, we find that $\delta_m(f_i) = 0$. By Lemma 5.1, f_i belongs to R and $i = 0$.

Assertion 2: $f \neq f_0$. Assume on the contrary that $f = f_0$. Since $d_1 = \partial/\partial x_1$, we have $d_1(f) = 0$. As $\Delta_m = x_1^m d_1 + x_2^m d_2$ and $\Delta_m(f) = 0$, we get $d_1(f) = d_2(f) = 0$, hence a contradiction.

Assertion 3: Let k be the smallest positive index such that $f_k \neq 0$. Then $2 \leq k \leq m'_0 + 2$. If we consider the terms of degree $m + i - 1$ in the equation $\Delta_m(f) = 0$, we obtain

$$\delta_m(f_i) + x_1 x_2^m \{\partial_0(f_{i-2}) + \partial_1(f_{i-3}) + \dots + \partial_{i-2}(f_0)\} = 0$$

for any $i > 1$, and $\delta_m(f_1) = 0$. By Lemma 5.1, we have $f_1 = 0$ and $k \geq 2$. Assume on the contrary that $k > m'_0 + 2$. Then for any $0 < i < k$, we have $f_i = 0$. This implies that $\partial_i(f_0) = 0$ for any $i < k - 2$, and we have in particular

$$\partial_0(f_0) = \partial_1(f_0) = \dots = \partial_{m'_0}(f_0) = 0.$$

By assumption, $\partial_0, \dots, \partial_{m'_0}$ span the module M generated by the ∂_l . For any nonnegative integer l , there exist some formal elements $b_{l,\beta}$ such that

$$\partial_l = \sum_{\beta=0}^{m'_0} b_{l,\beta} \partial_\beta.$$

This implies that $\partial_l(f_0) = 0$ for any l . In particular f_0 is annihilated by d_2 . Since $d_1(f_0) = 0$, f_0 belongs to $\ker \Delta_m$. So $(f - f_0)$ lies in $\ker \Delta_m$ and has no homogeneous part of degree 0. By our first assertion, $(f - f_0)$ needs to be equal to 0. But that contradicts our second assertion.

Assertion 4: The final contradiction. With k as in Assertion 3, if we consider the terms of degree $m + k - 1$ in the equation $\Delta_m(f) = 0$, we obtain

$$\delta_m(f_k) + x_1 x_2^m \partial_{k-2}(f_0) = 0$$

because $f_1 = \dots = f_{k-1} = 0$. By the previous assertion, $k \leq m'_0 + 2$ and $\partial_{k-2}(f_0)$ is an homogeneous polynomial of degree $\leq m'_0$. In particular, we find

$$m \geq m_0 \geq m'_0 + 4 \geq \deg(\partial_{k-2}(f_0)) + 4.$$

By applying Lemma 5.2 to the relation $\delta_m(f_k) + x_1 x_2^m \partial_{k-2}(f_0) = 0$, we get that $f_k = 0$, which is impossible by the third assertion. \square

Theorem 1.3 is a direct consequence of this proposition. Indeed if $m \geq m_0$ and f is an element of A such that $\Delta_m(f) = 0$, then $d_1(f) = d_2(f) = 0$ as elements of $L[[t_1, \dots, t_n]]$. Since A is a subring of $L[[t_1, \dots, t_n]]$, f belongs to k and the result follows.

7. A few consequences and an example

In this section, we are going to derive some consequences of Theorem 1.3. In particular, we will give another proof of Nowicki's theorem.

7.1. Proof of Theorem 1.6

Let k be an algebraically closed field of characteristic zero. Let A be a k -algebra of finite type with no zero divisors. Let \mathcal{F} be a family of k -derivations on A , and denote by $M_{\mathcal{F}}$ the A -module spanned by \mathcal{F} . Recall that $M_{\mathcal{F}, \min}$ is the set of \mathcal{F} -minimal derivations, i.e. the subset of $M_{\mathcal{F}}$ formed by the k -derivations d such that

$$\ker \mathcal{F} = \ker d.$$

Let us prove that $M_{\mathcal{F}, \min}$ is a non-empty residual subset of $M_{\mathcal{F}}$. The non-emptiness is guaranteed by Theorem 1.3. For residuality, let F be any finite k -vector subspace of $M_{\mathcal{F}}$. Since A is a finite k -algebra, its dimension is at most countable. So the space $A/\ker \mathcal{F}$ admits a filtration $\{F_n\}_{n \in \mathbb{N}}$, where every F_n is finite dimensional, i.e.

$$\frac{A}{\ker \mathcal{F}} = \bigcup_{n \in \mathbb{N}} F_n \quad \text{and} \quad F_0 \subseteq F_1 \subseteq \dots \subseteq F_n \subseteq \dots.$$

Consider the set Σ_n defined as

$$\Sigma_n = \{(d, f) \in F \times \mathbb{P}(F_n), d(f) = 0\}.$$

Note that Σ is an algebraic subset of $F \times \mathbb{P}(F_n)$ by construction. If $\Pi : F \times \mathbb{P}(F_n) \rightarrow F$ denotes the standard projection on F , we find

$$\Pi(\Sigma_n) = \{d \in F, \exists d \in \mathbb{P}(F_n), d(f) = 0\}.$$

In particular, a k -derivation d in F is \mathcal{F} -minimal if and only if it does not belong to the union $\bigcup_{n \in \mathbb{N}} \Pi(\Sigma_n)$. In other words we have

$$M_{\mathcal{F}, \min} \cap F = F - \bigcup_{n \in \mathbb{N}} \Pi(\Sigma_n) = \bigcap_{n \in \mathbb{N}} (F - \Pi(\Sigma_n)).$$

But every Σ_n is closed in $F \times \mathbb{P}(F_n)$ for the Zariski topology. Since $\mathbb{P}(F_n)$ is projective, it is a complete variety and the projection Π is closed. Therefore every set $\Pi(\Sigma_n)$ is closed in

F , and $M_{\mathcal{F}, \min}$ intersects F in a countable intersection of Zariski open sets. By definition, $M_{\mathcal{F}, \min}$ is residual in $M_{\mathcal{F}}$.

7.2. Proof of Nowicki's result

In this subsection, we will use Theorem 1.3 in order to establish Theorem 1.2. The proof is based on the following lemma.

Lemma 7.1. *Let A be a finite k -algebra with no-zero divisors. Let B be a sub-algebra of A and let d_B be a k -derivation on B . Then there exists a non-zero element a of A and a k -derivation d_A on A such that $d_A = ad_B$ on B .*

Proof. Let x_1, \dots, x_n be a set of generators of A as a k -algebra. If $K = Fr(B)$ and $L = Fr(A)$, then we have $L = K(x_1, \dots, x_n)$. Consider the following chain of fields:

$$K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_n, \quad \text{where } K_i = K(x_1, \dots, x_i).$$

By construction we have $K_{i+1} = K_i(x_{i+1})$. Assume there exists a k -derivation ∂_i on K_i . Then there exists a k -derivation ∂_{i+1} on K_{i+1} that extends the derivation ∂_i (if x_i is transcendental over K_i , see [6, p. 10] and if x_i is algebraic over K_i , see [7, p. 14]). By a finite induction, we can extend the k -derivation $d_B = \partial_0$ of $K = K_0$ into a k -derivation $d_L = \partial_n$ on $L = K_n$. For any index i , there exists a non-zero element a_i of A such that $a_i d_L(x_i)$ belongs to A . If $a = a_1 \dots a_n$, then $ad_L(x)$ belongs to A for any x in A . The derivation $d_A = ad_L$ maps A into A and thus defines a k -derivation on A . By construction we have $d_A = ad_B$ on B . \square

Proof of Theorem 1.2. Let A be a finite k -algebra with no zero divisors over a field k of characteristic zero. Let d be a k -derivation on A and set $B = \ker d$. By Leibniz rule, B needs to be a k -algebra. Let a be any element of A that is algebraic over B . Let $P(t) = b_0 + \dots + b_n t^n$ be a polynomial of $B[t]$ of minimal degree such that $P(a) = 0$. By derivation, we get

$$d(P(a)) = \left(\sum_{k=1}^n k a^{k-1} \right) d(a) = P'(a) d(A) = 0.$$

By minimality of the degree, we find $P'(a) \neq 0$ and $d(A) = 0$. So a belongs to B and B is algebraically closed in A .

Conversely let B be a sub-algebra of A and assume that B is algebraically closed in A . For any x outside B , denote by B_x the algebra $B[x]$ and consider the k -derivation $d_x = d/dx$ on B_x . Since x lies outside B , x is transcendental over B and d_x is well-defined. By Lemma 7.1, there exist a non-zero element a_x of A and a k -derivation ∂_x on A such that $\partial_x = a_x d_x$. In particular $\partial_x(x) = a_x d_x(x) = a_x \neq 0$. Consider the family $\mathcal{F} = \{\partial_x\}_{x \in A-B}$ of k -derivations on A . By construction, we have

$$\ker \mathcal{F} = B.$$

By Theorem 1.3, there exists an A -linear combination d of the ∂_x such that $\ker d = B$. In particular B is the kernel of a k -derivation and the result follows. \square

7.3. An example

In this subsection, we will illustrate the notion of residuality in an example. Consider the algebra $A = \mathbb{C}[x, y]$ provided with the following couple of \mathbb{C} -derivations:

$$\mathcal{F} = \{d_1, d_2\} = \left\{ x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y} \right\}.$$

Let F be the vector space spanned by d_1, d_2 . Then it is easy to check that an element $\lambda_1 d_1 + \lambda_2 d_2$ of F admits a first integral if and only if either $\lambda_2 = 0$ or λ_1/λ_2 is a non-positive rational number. In this latter case, if $\lambda_1/\lambda_2 = -p/q$, where $p \geq 0$, $q > 0$ are coprime integers, then every first integral of $\lambda_1 d_1 + \lambda_2 d_2$ is a polynomial function of the expression:

$$f_{p,q}(x, y) = x^q y^p.$$

Let \mathcal{D} be the union of the coordinate axis of \mathbb{C}^2 and of the lines of rational negative slopes. Then \mathcal{D} is a countable union of complex lines in \mathbb{C}^2 . If we identify d_1, d_2 with the canonical basis of \mathbb{C}^2 , then we have

$$F \cap M_{\mathcal{F}, \min} = F - \mathcal{D}$$

and this latter set turns out to be residual, since it is a countable intersection of Zariski open sets of $F \simeq \mathbb{C}^2$. Residuality occurs as we cannot find a finite dimensional space in $\mathbb{C}[x, y]$ containing all the first integrals of minimal degree of elements of F . If it were possible, then as in Theorem 1.6, we could prove that $F \cap M_{\mathcal{F}, \min}$ would be a finite intersection of Zariski open sets, hence an open set and this is not the case.

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